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Geometrization of Two-Phase Magnetofluid Flows in a Rotating System

C. Thakur[1] and R. B. Mishra[1]

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By employing a nonholonomic description of the governing equations, the complex lamellar motion and Beltrami motion of steady-rotating, viscous, incompressible, perfectly conducting two-phase fluid flows in the presence of a magnetic field are discussed and some results of physical importance are derived.

1. INTRODUCTION

Multiphase fluid phenomena are of extreme importance in various fields of science and technology, such as geophysics, nuclear engineering, and chemical engineering. In recent years, considerable attention has been given to the study of the multiphase fluid flow system in a nonrotating or rotating frame of reference. The multiphase fluid systems are concerned with the motion of a liquid or gas containing immiscible inert particles. Of all multiphase fluid systems observed in nature, blood flow, flow in rocket tubes, dust in gas cooling systems to enhance the heat transfer movement of inert particles in atmospheres, and sand or other suspended particles in seawater are the most common examples of multiphase fluid systems. Studies of these systems are mathematically interesting and physically useful. The presence of particles in a homogeneous fluid makes the dynamical study of the flow problem quite complicated. However, these problems are usually investigated under various simplifying assumptions.

Saffman (1962) formulated the equations of motion of a dusty fluid which is represented in terms of a large number density $N(x, t)$ of very small, spherical, inert particles whose volume concentration is small enough to be neglected. It is assumed that the density of the dust particles is large compared with the fluid density so that the mass concentration of the

[1]Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi 221005, India.

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particles is an appreciable fraction of unity. In this formulation, Saffman also assumed that the treatment of individual particles and the fluid remains valid.

Using the model of Saffman, several authors (e.g., Michael and Miller, 1966; Michael, 1968; Liu, 1967; Healy and Yang, 1972; Singh *et al.* 1984) have investigated various aspects of hydrodynamic and hydromagnetic two-phase fluid flows in a nonrotating system. On the other hand, the simultaneous influence of rotation and external magnetic field on electrically conducting two-phase fluid flow systems seems to be dynamically important and physically useful. In spite of the above work, the dynamics of the two-phase conducting fluid flow in a rotating system has hardly received any attention.

The intrinsic description of a three-dimensional vector field in terms of nonholonomic coordinates was first introduced by Vranceanu (1936) and was employed by Marris and Passman (1969) to describe some kinematic properties of fluid flows. This geometry was applied by Rogers and Kingston (1974), Singh and Babu (1983), Singh and Singh (1984), and Singh *et al.* (1984) in the study of MFD flows.

The main objective of this paper is the study of the geometry of two-phase fluid flow in a rotating coordinate system. We have decomposed the governing equations along the vortex line triad. The complex lamellar motion and Beltrami motion are studied and the conditions are obtained for which the vortex lines are straight lines normal to Maxwellian surface which are minimal surface. The variation of total energy along the vortex lines is also discussed.

2. BASIC EQUATIONS

The basic equations governing the steady motion of an incompressible, viscous, two-phase fluid flow with infinite electrical conductivity in a rotating coordinate system under an external magnetic field are given by

$$\operatorname{div} \mathbf{u} = 0 \tag{1}$$

$$\rho[(\mathbf{u} \cdot \operatorname{grad})\mathbf{u} + 2\boldsymbol{\omega} \times \mathbf{u}] = -\operatorname{grad} p^* + \eta\, \nabla^2 \mathbf{u} + \mu \operatorname{curl} \mathbf{H} \times \mathbf{H} + KN(\mathbf{v} - \mathbf{u}) \tag{2}$$

$$\operatorname{curl}(\mathbf{u} \times \mathbf{H}) = \mathbf{0} \tag{3}$$

$$\operatorname{div}(N\mathbf{v}) = 0 \tag{4}$$

$$m[(\mathbf{v} \cdot \nabla)\mathbf{v} + 2\boldsymbol{\omega} \times \mathbf{v}] = K(\mathbf{u} - \mathbf{v}) \tag{5}$$

$$\operatorname{div} \mathbf{H} = 0 \tag{6}$$

where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{H}$, P^*, ρ and η are, respectively, the fluid velocity vector, the dust velocity vector, the magnetic field vector, the modified fluid pressure, the fluid density, and the kinematic coefficient of viscosity; m is the mass of each dust particle, N is the number density of dust particles, and K is the Stokes resistance coefficient for the dust particles.

The situation for which the velocities of fluid and dust particles are everywhere parallel is defined as

$$\mathbf{v} = \frac{\alpha}{N}\mathbf{u} \tag{7}$$

where α is some scalar satisfying

$$\mathbf{u} \cdot \text{grad}\, \alpha = 0 \tag{8}$$

which implies that α is constant on the fluid streamlines. Using the vector identities,

$$(\mathbf{v} \cdot \text{grad})\mathbf{v} = \tfrac{1}{2}\,\text{grad}(\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times \text{curl}\, \mathbf{v}$$

$$\text{curl curl}\, \mathbf{v} = \text{grad div}\, \mathbf{v} - \nabla^2 \mathbf{v}$$

along with the substitution curl $\mathbf{v} = \boldsymbol{\xi}$ in the momentum equation (2), we get

$$\nabla B = \rho\{\mathbf{u} \times (2\boldsymbol{\omega} + \boldsymbol{\xi})\} + \mu \mathbf{H} \cdot \text{grad}\, \mathbf{H} - \eta\, \text{curl}\, \boldsymbol{\xi} + K(\alpha/N - 1)\mathbf{u} \tag{9}$$

which is the Bernoulli equation; here $B = \frac{1}{2}\rho u^2 + P^* + \frac{1}{2}\mu H^2$, u is the magnitude of the velocity, and H is the magnetic field strength.

3. NONHOLONOMIC GEOMETRIC RESULTS

Let us introduce the orthonormal basis $\mathbf{s}$, $\mathbf{n}$, $\mathbf{b}$ along the vertex lines, where $\mathbf{s}$ is the unit tangent; $\mathbf{n}$ is the unit principal normal; and $\mathbf{b}$ is the unit binormal. Now we state some geometric results due to Rogers and Kingston (1974) in terms of eight parameters k, τ, Ω_s, Ω_n, div $\mathbf{n}$, div $\mathbf{b}$, $\theta_{ns} = \mathbf{n} \cdot \text{grad}\, \mathbf{s} \cdot \mathbf{n}$, and $\theta_{bs} = \mathbf{b} \cdot \text{grad}\, \mathbf{s} \cdot \mathbf{b}$:

$$\frac{\delta \mathbf{s}}{\delta s} = k\mathbf{n} \tag{10}$$

$$\frac{\delta \mathbf{n}}{\delta s} = -k\mathbf{s} + \tau \mathbf{b} \tag{11}$$

$$\frac{\delta \mathbf{b}}{\delta s} = -\tau \mathbf{n} \tag{12}$$

$$\frac{\delta \mathbf{s}}{\delta n} = \theta_{ns}\mathbf{n} - (\tau + \Omega_n - \Omega_s)\mathbf{b} \tag{13}$$

$$\frac{\delta \mathbf{n}}{\delta n} = \theta_{ns}\mathbf{s} - \text{div}\, \mathbf{b}\mathbf{b} \tag{14}$$

$$\frac{\delta \mathbf{b}}{\delta n} = (\tau + \Omega_n - \Omega_s)\mathbf{s} + \text{div } \mathbf{bn} \quad (15)$$

$$\frac{\delta \mathbf{s}}{\delta b} = -(\tau + \Omega_n)\mathbf{n} + \theta_{bs}\mathbf{b} \quad (16)$$

$$\frac{\delta \mathbf{n}}{\delta b} = (\tau + \Omega_n)\mathbf{s} + (k + \text{div } \mathbf{n})\mathbf{b} \quad (17)$$

$$\frac{\delta \mathbf{b}}{\delta b} = \theta_{bs}\mathbf{s} - (k + \text{div } \mathbf{n})\mathbf{n} \quad (18)$$

where $\delta/\delta s = \mathbf{s} \cdot \nabla$ is the intrinsic derivative along **s** lines, k is the curvature, and τ is the torsion of the **s** lines. $\Omega_s = \mathbf{s} \cdot \text{curl } \mathbf{s}$ and $\Omega_n = \mathbf{n} \cdot \text{curl } \mathbf{n}$ are the abnormalities of **s** lines and **n** lines, respectively. The relations (10)-(12) are the Serret-Frenent formulas.

It should be noted that here we employ the parameter Ω_n instead of $\psi = \Omega_b - \Omega_s$ as employed by Marris and Passman (1969), and Ω_b is the abnormality of **b** lines expressed in terms of the other parameters by the relation

$$\Omega_b = -2\tau + \Omega_s + \Omega_n \quad (19)$$

The further results due to Marris and Passman are

$$\text{curl } \mathbf{s} = \Omega_s \mathbf{s} + k\mathbf{b} \quad (20)$$

$$\text{curl } \mathbf{n} = -\text{div } \mathbf{bs} + \Omega_n \mathbf{n} + \theta_{ns}\mathbf{b} \quad (21)$$

$$\text{curl } \mathbf{b} = (k + \text{div } \mathbf{n})\mathbf{s} - \theta_{ns}\mathbf{n} + \Omega_b \mathbf{b} \quad (22)$$

$$\text{div } \mathbf{s} = \theta_{ns} + \theta_{bs} \quad (23)$$

4. THE GEOMETRY OF VORTEX LINES

The vectors $\boldsymbol{\xi}, \mathbf{u}, \mathbf{H}, \boldsymbol{\omega}$ can be expressed as

$$\boldsymbol{\xi} = \xi \mathbf{s} \quad (24)$$

$$\mathbf{u} = u_s \mathbf{s} + u_n \mathbf{n} + u_b \mathbf{b} \quad (25)$$

$$2\boldsymbol{\omega} = \omega_s \mathbf{s} + \omega_n \mathbf{n} + \omega_b \mathbf{b} \quad (26)$$

$$\mathbf{H} = h_s \mathbf{s} + h_n \mathbf{n} + h_b \mathbf{b} \quad (27)$$

where ξ is the magnitude of vorticity, u_s, u_n and u_b are the velocity components; ω_s, ω_n and ω_b are the angular velocity components; and h_s, h_n and h_b are the magnetic field vector components in the directions **s**, **n** and **b**, respectively. Operating with curl on (25) and using equations (20)-(22) and (24), we obtain

$$\xi = \frac{\delta u_b}{\delta n} - \frac{\delta u_n}{\delta b} + u_s \Omega_s - u_n \operatorname{div} \mathbf{b} + u_b (k + \operatorname{div} \mathbf{n}) \tag{28}$$

$$\frac{\delta u_s}{\delta b} - \frac{\delta u_b}{\delta s} + u_n \Omega_n - u_b \theta_{bs} = 0 \tag{29}$$

$$\frac{\delta u_n}{\delta s} - \frac{\delta u_s}{\delta n} + k u_s + \theta_{ns} u_n + \Omega_b u_b = 0 \tag{30}$$

These are the geometric conditions satisfied by the geometric parameters of vortex-line triad and are independent of the nature of the fluid.

Using the solenoidal property of the vorticity vector, we get

$$\frac{\delta}{\delta s} \ln \xi + \operatorname{div} \mathbf{s} = 0 \tag{31}$$

Thus, by (31) and (23) we conclude the following theorem.

Theorem 1. In steady, rotating, two-phase magnetofluid flows, the magnitude of vorticity remains constant along vortex lines if and only if the deformations of vortex lines along their principal normal and binormal vanish. The converse is also true.

Using equations (24)-(27) and the geometric results of the preceding section in equation of motion (9), field equation (3), and in (5), we get the decompositions

$$\begin{aligned}\frac{\delta B}{\delta s} = \mu \bigg[& h_s \frac{\delta h_s}{\delta s} + h_n \frac{\delta h_s}{\delta n} + h_b \frac{\delta h_s}{\delta b} - h_s h_n k \\ & + h_n h_b (2\tau + 2\Omega_n - \Omega_s) - h_n^2 \theta_{ns} - h_b^2 \theta_{bs} \bigg] \\ & + \rho (u_n \omega_b - u_b \omega_n) - \eta \xi \Omega_s + KN\left(\frac{\alpha}{N} - 1\right) u_s \end{aligned} \tag{32}$$

$$\begin{aligned}\frac{\delta B}{\delta b} = \mu \bigg[& h_s \frac{\delta h_n}{\delta s} + h_n \frac{\delta h_n}{\delta n} + h_b \frac{\delta h_n}{\delta b} + h_n h_b \operatorname{div} b \\ & + h_n h_s \theta_{ns} - h_b h_s (2\tau + \Omega_n) + h_s^2 k \bigg] \\ & - \eta \frac{\delta \xi}{\delta b} - h_b^2 (k + \operatorname{div} \mathbf{n}) + \rho \xi u_b \\ & + \rho (u_b \omega_s - u_s \omega_b) + KN\left(\frac{\alpha}{N} - 1\right) u_n \end{aligned} \tag{33}$$

$$\frac{\delta B}{\delta b}=\mu\left[h_s\frac{\delta h_b}{\delta s}+\frac{h_n\delta h_b}{\delta n}+h_b\frac{\delta h_b}{\delta b}+h_bh_s\theta_{bs}\right.$$
$$+h_bh_n(k+\operatorname{div}\mathbf{n})-h_nh_s(\Omega_n-\Omega_s)$$
$$\left.-h_n^2\operatorname{div}\mathbf{b}\right]-\rho\xi u_n+\eta\left(\frac{\delta\xi}{\delta n}-k\xi\right)$$
$$+\rho(u_s\omega_n-u_n\omega_s)+KN\left(\frac{\alpha}{N}-1\right)u_b \tag{34}$$

$$\frac{\delta A_b}{\delta n}-\frac{\delta A_n}{\delta b}+A_s\Omega_s-A_n\operatorname{div}\mathbf{b}+A_b(k+\operatorname{div}\mathbf{n})=0 \tag{35}$$

$$\frac{\delta A_s}{\delta b}-\frac{\delta A_b}{\delta s}+A_n\Omega_n-A_b\theta_{bs}=0 \tag{36}$$

$$\frac{\delta A_n}{\delta s}-\frac{\delta A_s}{\delta n}+A_b\Omega_b+A_sk+A_n\theta_{ns}=0 \tag{37}$$

where

$$A_s=u_nh_b-u_bh_n,\qquad A_n=u_bh_s-u_sh_b,\qquad A_b=u_sh_n-u_nh_s$$

$$\frac{\alpha^2}{N}u_s\frac{\delta}{\delta s}\left(\frac{u_s}{N}\right)+\frac{\alpha^2}{N}u_n\frac{\delta}{\delta n}\left(\frac{u_s}{N}\right)+\frac{\alpha^2}{N}u_b\frac{\delta}{\delta b}\left(\frac{u_s}{N}\right)$$
$$-\left(\frac{\alpha}{N}\right)^2u_su_nk+\left(\frac{\alpha}{N}\right)^2u_nu_b(2\tau-\Omega_s+2\Omega_n)$$
$$+\left(\frac{\alpha u_n}{N}\right)^2\theta_{ns}-\left(\frac{\alpha u_b}{N}\right)^2\theta_{bs}+\frac{\alpha}{N}(\omega_nu_b-\omega_bu_n)$$
$$=\frac{K}{m}\left(1-\frac{\alpha}{N}\right)u_s \tag{38}$$

$$\frac{\alpha^2}{N}u_s\frac{\delta}{\delta s}\left(\frac{u_n}{N}\right)+\frac{\alpha^2}{N}u_n\frac{\delta}{\delta n}\left(\frac{u_n}{N}\right)+\frac{\alpha^2}{N}u_b\frac{\delta}{\delta b}\left(\frac{u_n}{N}\right)$$
$$+\left(\frac{\alpha}{N}\right)^2u_su_n\theta_{ns}+\left(\frac{\alpha}{N}\right)^2u_nu_b\operatorname{div}\mathbf{b}+\left(\frac{\alpha}{N}u_s\right)^2k$$
$$-\left(\frac{\alpha}{N}\right)^2u_su_b(2\tau+\Omega_n)-\left(\frac{\alpha u_b}{N}\right)^2(k+\operatorname{div}\mathbf{n})$$
$$+\frac{\alpha}{N}(u_s\omega_b-u_b\omega_s)=\frac{K}{m}\left(1-\frac{\alpha}{N}\right)u_n \tag{39}$$

$$\frac{\alpha^2}{N}u_s\frac{\delta}{\delta s}\left(\frac{u_b}{N}\right)+\frac{\alpha^2}{N}u_n\frac{\delta}{\delta n}\left(\frac{u_b}{N}\right)+\frac{\alpha^2}{N}u_b\frac{\delta}{\delta b}\left(\frac{u_b}{N}\right)$$
$$+\left(\frac{\alpha}{N}\right)^2 u_s u_n(\Omega_s-\Omega_n)+\left(\frac{\alpha}{N}\right)^2 u_n u_b(k+\operatorname{div}\mathbf{b})$$
$$+\left(\frac{\alpha}{N}\right)^2 u_s u_b \theta_{bs}-\left(\frac{\alpha u_n}{N}\right)^2 \operatorname{div}\mathbf{b}$$
$$+\frac{\alpha}{N}(u_n\omega_s-u_s\omega_n)=\frac{K}{m}\left(1-\frac{\alpha}{N}\right)u_b \tag{40}$$

5. COMPLEX LAMELLAR MOTION

The motion is complex lamellar if the velocity $\mathbf{u}$ is such that (Marris and Passman, 1969)

$$\mathbf{u}\cdot\operatorname{curl}\mathbf{u}=0,\quad \text{i.e.,}\quad \mathbf{u}\cdot\boldsymbol{\xi}=0 \tag{41}$$

The condition (41) implies that $u_s=0$. Let us assume that the magnetic field lines are in the normal planes of the vortex lines, i.e., $h_s=0$. Then equations (35)-(37) are reduced to the form

$$\Omega_s A_s=0 \tag{42}$$

$$\delta A_s/\delta b=0 \tag{43}$$

$$\delta \ln A_s/\delta n=k \tag{44}$$

If the streamlines and the magnetic field lines are nonparallel, $A_s\neq 0$, and hence, from equation (42), we have

$$\Omega_s=0 \tag{45}$$

which ensures the existence of a family of surfaces orthogonal to the vortex lines. These surfaces contain the streamlines and the magnetic field lines, and hence they are Maxwellian surfaces, $-\operatorname{div}\mathbf{s}$ is the mean curvature of such surfaces. Therefore, from the above discussion and equation (31), we conclude the following theorem.

Theorem 2. For the complex lamellar motion of two-phase magnetofluid flow in a rotating coordinate system, where the magnetic lines are in the normal plane of the vortex lines, (i) the vortex lines are orthogonal to the Maxwellian surface, and (ii) the magnitude of the vorticity is uniform along vortex lines if and only if the Maxwellian surfaces are minimal surfaces.

Suryanarayan (1972) has shown that $Hu \sin \alpha^*$ is constant on the Maxwellian surfaces, where α^* is the angle between the streamlines and the magnetic field lines. But $Hu \sin \alpha^* = |\mathbf{u} \times \mathbf{H}|$, which is equal to A_s. Therefore, A_s is constant on the Maxwellian surfaces. Since **n** lines and **b** lines lie on these surfaces, then, from (44), $k=0$. Thus, we conclude the following theorem.

Theorem 3. For the complex lamellar motion of two-phase rotating magnetofluid flows where the magnetic field lines lie in the normal planes of the vortex lines, the vortex lines are straight lines.

If the magnetic field lines and streamlines are along the binormals of the vortex lines, then from equations (32), (38), and (40) we have

$$\frac{\delta B}{\delta s} = -\mu h_b^2 \theta_{bs} - \rho u_b \omega_n \tag{46}$$

$$-\left(\frac{\alpha u_b}{N}\right)^2 \theta_{bs} + \frac{\alpha}{N}(\omega_n u_b) = 0 \tag{47}$$

and

$$\frac{\alpha^2}{N} u_b \frac{\delta}{\delta b}\left(\frac{u_b}{N}\right) = \frac{K}{m}\left(1 - \frac{\alpha}{N}\right) u_b \tag{48}$$

The second terms on the right-hand side of (47) and (48) are the components of the coriolis force along the vortex lines. Thus, we have Theorem 4.

Theorem 4. For complex lamellar motion of two-phase magnetofluid flows in a rotating coordinate system where the streamlines and magnetic field lines are along the binormals of vortex lines, the total energy is constant along the vortex lines if and only if the component of the coriolis force along the vortex lines is given by $(H^2/\rho)\theta_{bs}$.

Let us assume that the deformation of the vortex tube along the magnetic field lines is zero; then we have the following results as a special case of Theorem 4.

Corollary 1. The total energy is constant along the vortex lines if and only if the component of coriolis force along vertex lines is zero.

If u_b/N is constant along the binormals of vortex lines, equation (48) gives

$$\alpha/N = 1 \tag{49}$$

Thus, from (49) and (7), we conclude the proof of the following Theorem 5.

Theorem 5. For complex lamellar motion of two-phase, rotating, magnetofluid flows, if the ratio of fluid velocity and number density of dust particles is constant along binormals of the vortex lines, the magnitude of the fluid velocity is equal to the velocity of the dust.

6. BELTRAMI MOTION

The motion is Beltrami if and only if (Truesdell)

$$\mathbf{u}\times\operatorname{curl}\mathbf{u}=0, \quad \text{i.e.,} \quad \mathbf{u}\times\boldsymbol{\xi}=0 \tag{50}$$

The condition (50) implies $u_n=u_b=0$. Let us assume that the magnetic field lines lie in the rectifying planes of the vortex lines; then $h_n=0$, and hence equations (35)-(37) are reduced to the form

$$\delta A_n/\delta b+A_n \operatorname{div}\mathbf{b}=0 \tag{51}$$

$$A_n\Omega_n=0 \tag{52}$$

$$\delta \ln A_n/\delta s+\theta_{ns}=0 \tag{53}$$

Now we consider equation (52) if the streamlines and magnetic field lines are not parallel, $A_n\neq 0$; hence

$$\Omega_n=0 \tag{54}$$

Theorem 6. For Beltrami motion of two-phase, rotating magnetofluid flows where the magnetic field lines lie in the rectifying planes of the vortex lines, the family of Maxwellian surfaces is a normal congruence, the vortex lines are the geodesics, and their binormals are geodesic parallels.

Suryanarayan has shown that $Hu \sin \alpha^*$ is constant on the Maxwellian surfaces, where α^* is the angle between the streamlines and magnetic field lines. Here $\alpha^*=\pi/2$; hence, from equation (53), we have

$$\theta_{ns}=0 \tag{55}$$

Let us assume that the vorticity is constant along the vortex lines; then, using (55) in equation (1), we have

$$\theta_{bs}=0 \tag{56}$$

This is the geodesic curvature of b lines, which vanishes. Vector fields satisfying the latter condition together with (54) are said to be flat. Hence the following theorem holds.

Theorem 7. For Beltrami motion of two-phase, rotating magnetofluid flows, where the magnetic field lines lie in the rectifying planes of the vortex lines and the magnitude of velocity remains uniform along the vortex lines, the family of Maxwellian surfaces is a normal congruence of developables.

If the magnetic field lines are along the binormals of the vortex lines, we have, imposing the flat field restriction (54) and (56) in equation (32),

$$\delta B/\delta s = -\eta\xi\Omega_s + KN(\alpha/N - 1)u_s \tag{57}$$

Also from equation (38), we have

$$\frac{\alpha^2}{N} u_s \frac{\delta}{\delta s}\left(\frac{u_s}{N}\right) = \frac{K}{m}\left(1 - \frac{\alpha}{N}\right) u_s \tag{58}$$

If u_s/N is constant along the tangent of vortex lines, from equation (58), we have

$$\alpha/N = 1 \tag{59}$$

From equations (59) and (7) we conclude the following theorem.

Theorem 8. For Beltrami motion of two-phase, rotating magnetofluid flows, if the ratio of the fluid velocity and number density of dust particles is constant along the tangent of the vortex lines, the magnitude of the fluid velocity is equal to the velocity of the dust.

Using equations (57) and (59), we get

$$\delta B/\delta s = -\eta\xi\Omega_s \tag{60}$$

We conclude the following theorem.

Theorem 9. For Beltrami motion of two-phase, rotating magnetofluid flows where magnetic field lines are along the binormals of the vortex lines with flat field restrictions, the Bernoulli surfaces contain the vortex lines if and only if the magnitude of fluid velocity is equal to the velocity of the dust.

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